Maximum allowed angular errors for positioning mirrors in Advanced Virgo OMC

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1 Introduction

We study in this paper the propagation of an optical beam inside the Output Mode Cleaner (OMC) cavity in order to test the tolerancing of construction specifications. The main goal of this problem is to check if the beam stay inside polished mirrors during the propagation assuming random angular errors for mirror positions. The diameters of mirrors, assume to be 5mm, and the maximum allowed angular error, which we would determine, constitute the set of two relevant parameters in this problem if we exclude configuration parameters of the OMC. To solve the problem we will progress in two stpng, first we will establish the equations of the beam propagation inside the cavity using the geometrical optics and second we will do the numerical resolution with a C++ code of those equations and find the maximum allowed angular error.

2 Equations

2.1 Introduction to the mathematical problem

2.1.1 Configuration of the cavity

We can find characteristics of the OMC in the Advanced Virgo Technical Design Report. We put a number for each mirror following the figure 1 and all variables in relation of a mirror will take the same subscript number. Angles are oriented in the counterclockwise. The axis going from the center $\mathcal{O}_1$ to the center $\mathcal{O}_2$ with the associated base vector $\mathbf{e_x}$ will be the reference axis for all of this study. The origin of frames will be $\mathcal{O}_1$. We will use unitary vectors $\mathbf{u_i}$ parallel to the mirror i (see Fig. 1). The pair of vectors $(\mathbf{e_x}, \mathbf{u_i})$ make a base which is not orthogonal. Finally, mirrors 1 and 4 are plane and mirrors 2 and 3 are spherical.

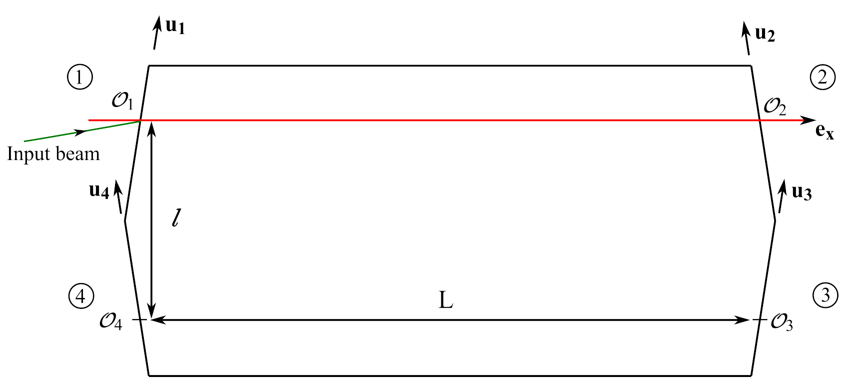

Figure 1: Configuration of the cavity

We will consider beams only inside the cavity, input and output beams are deduced from corresponding inside beam with the refraction law.

2.1.2 Position of mirrors and beams

The position of mirrors and beams is done in the following way (see figure. 2):

- γ_i is the angle between $\mathbf{e_x}$ and $\mathbf{u_i}$ which represent the angular position of the mirror i from the x axis.
- α_i is the angle between an orthogonal axis to the mirror i and the incident beam.
- α_{ir} is the angle between an orthogonal axis to the mirror i and the reflected beam.
- L_i is the distance between the center of the mirror i and the impact point of the beam.

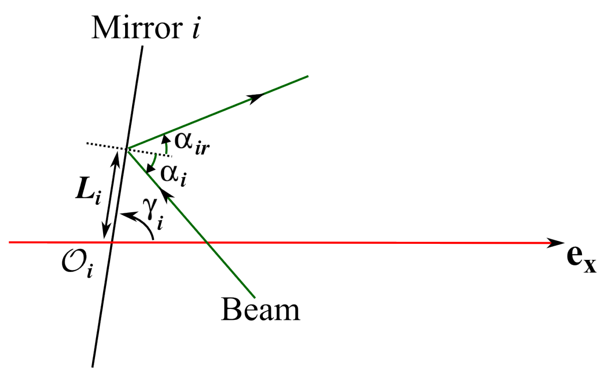

Figure 2: Variables of position for mirrors and beams

Coordinates of centers $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, $\mathcal{O}_4$ are respectively in their associated frame $\left((0,0),(L,0),(L+\frac{l}{\tan(\gamma_3)},-\frac{l}{\sin(\gamma_3)}),(\frac{l}{\tan(\gamma_4)},-\frac{l}{\sin(\gamma_4)})\right)$.

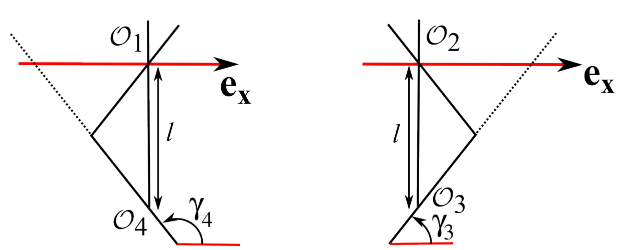

Figure 3: Positions of mirrors 3 and 4

2.2 Beam propagation with plane mirrors

2.2.1 Angles of incident beam as a function of the reflected beam

The angle α_j of the incident beam on the mirror j is calculated as a function of the angle α_{ir} by getting the angle θ. This angle represent the angular position of the beam from the x axis as shown on the figure 4.

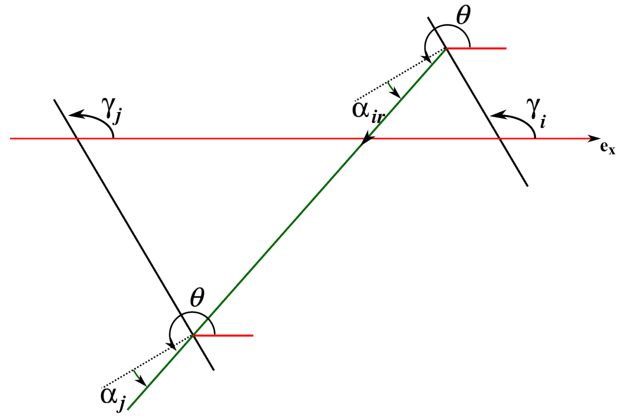

Figure 4: Beam positions

The angle θ is:

$$\theta = \gamma_i + \frac{\pi}{2} + \alpha_{ir} = \gamma_j + \frac{\pi}{2} + \alpha_j \tag{1}$$

Then we get:

$$\boxed{\alpha_j = \alpha_{ir} + \gamma_i - \gamma_j} \tag{2}$$

<u>**Comment :**</u> In our case of closed cavity, with the equation 2 we find the beam after reflections on the four mirrors has to respect the following equation:

$$\alpha_1 = -\alpha_{1r} + 2\left(\gamma_2 - \gamma_4 + \gamma_3 - \gamma_1\right) \tag{3}$$

We observe it exist a solution for the beam to come back with the same angular position using plane mirrors only in the particular case of $(\gamma_2 - \gamma_4 + \gamma_3 - \gamma_1) = 0$

2.2.2 Beam position from center of mirrors

We will specify in this section the position of the incident beam on the mirror j from the center $\mathcal{O}_j$ as a function of the reflected beam characteristics on the mirror i. We will represent the propagation of the beam from i to j with the vector $\mathbf{v_{ij}}$. In order to obtain searched parameters we will make projection of all quantities on base vectors $(\mathbf{e_x}, \mathbf{u_j})$.

First, we need to formulate $\mathbf{u_i}$ in the chosen base as:

$$\mathbf{u_i} = a\,\mathbf{e_x} + b\,\mathbf{u_j} \tag{4}$$

We know $\mathbf{u_i} \cdot \mathbf{e_x} = \cos(\gamma_i)$, $\mathbf{u_j} \cdot \mathbf{e_x} = \cos(\gamma_j)$ and $\mathbf{u_i} \cdot \mathbf{u_j} = \cos(\gamma_i - \gamma_j)$, from that scalar products we obtain :

$$\mathbf{u_i} = \frac{\sin(\gamma_j - \gamma_i)}{\sin(\gamma_j)}\,\mathbf{e_x} + \frac{\sin(\gamma_i)}{\sin(\gamma_j)}\,\mathbf{u_j} \tag{5}$$

Second, we formulate in the same way the vector $\mathbf{v_{ij}}$ as :

$$\mathbf{v_{ij}} = a\,\mathbf{e_x} + b\,\mathbf{u_j} \tag{6}$$

We need following scalar products :
$\mathbf{v_{ij}} \cdot \mathbf{e_x} = -\mu_{ij}\cos(\gamma_i + \frac{\pi}{2} + \alpha_{ir}) = \mu_{ij}\sin(\gamma_i + \alpha_{ir})$ and $\mathbf{v_{ij}} \cdot \mathbf{u_j} = \mu_{ij}\cos(\frac{\pi}{2} - \alpha_j) = \mu_{ij}\sin(\alpha_j)$. We specify the direction of propagation by using the constant μ_{ij} which is $\mu_{12} = \mu_{43} = 1$ and $\mu_{24} = \mu_{31} = -1$

After few trigonometrical calculus, we obtain:

$$\mathbf{v_{ij}} = \mu_{ij}\frac{\cos(\alpha_j)}{\sin(\gamma_j)}\,\mathbf{e_x} - \mu_{ij}\frac{\cos(\alpha_{ir} + \gamma_i)}{\sin(\gamma_j)}\,\mathbf{u_j} \tag{7}$$

Now, we will express the coordinates $(x_{\mathcal{O}_j}, L_j)$ in the frame $(\mathbf{e_x}, \mathbf{u_j})$ as a function of the coordinates $(x_{\mathcal{O}_i}, L_i)$. For this, we will write the equation of the line describing the propagation of the beam with the vector $\mathbf{v_{ij}}$, the origin $(x_{\mathcal{O}_i}, L_i)$ and the path parameter r:

$$(x_{\mathcal{O}_i}\mathbf{e_x} + L_i\mathbf{u_i}) + r\mathbf{v_{ij}} \tag{8}$$

The arriving impact point $\mathcal{T}$ on the mirror j correspond to the intersection between this line and the line representing the mirror j which has the coordinate $x_{\mathcal{O}_j}$ along the x axis. This intersection correspond to the following relation :

$$x_{\mathcal{O}_i}\mathbf{e_x} + L_i\mathbf{u_i} + r_{ij}\mathbf{v_{ij}} = x_{\mathcal{O}_j}\mathbf{e_x} + L_j\mathbf{u_j} \tag{9}$$

Since the vectors $\mathbf{e_x}$ and $\mathbf{u_j}$ form a base, the previous equation is verified when we have :

$$x_{\mathcal{O}_i} + L_i\frac{\sin(\gamma_j - \gamma_i)}{\sin(\gamma_j)} + r_{ij}\mu_{ij}\frac{\cos(\alpha_j)}{\sin(\gamma_j)} = x_{\mathcal{O}_j} \tag{10}$$

With the equation 10 we get r_{ij} :

$$r_{ij} = \mu_{ij}\left[x_{\mathcal{O}_j} - x_{\mathcal{O}_i} - L_i\frac{\sin(\gamma_j - \gamma_i)}{\sin(\gamma_j)}\right]\frac{\sin(\gamma_j)}{\cos(\alpha_j)} \tag{11}$$

Then we get the other coordinate L_j from r_{ij} :

$$L_j = L_i\frac{\sin(\gamma_i)}{\sin(\gamma_j)} - \mu_{ij}\left[x_{\mathcal{O}_j} - x_{\mathcal{O}_i} - L_i\frac{\sin(\gamma_j - \gamma_i)}{\sin(\gamma_j)}\right]\frac{\sin(\gamma_j)}{\cos(\alpha_j)}\mu_{ij}\frac{\cos(\alpha_{ir} + \gamma_i)}{\sin(\gamma_j)} \tag{12}$$

Let rewrite this expression as :

$$\boxed{L_j = L_i \frac{\sin(\gamma_i)}{\sin(\gamma_j)} - \left[x_{\mathcal{O}_j} - x_{\mathcal{O}_i} - L_i \frac{\sin(\gamma_j - \gamma_i)}{\sin(\gamma_j)}\right] \frac{\cos(\alpha_{ir} + \gamma_i)}{\cos(\alpha_j)}} \tag{13}$$

With equations 2 and 13 we can compute all reflexions inside the cavity taking care to put this additional equation for the reflection law:

$$\boxed{\alpha_{ir} = -\alpha_i} \tag{14}$$

2.3 Curved mirrors

2.3.1 Characterization of spherical mirrors

At this point, we will take into account the curvature radius of mirrors 2 and 3. First, we need to determine the coordinate position $(x_{\mathcal{K}_2}, L_{\mathcal{K}_2})$ and $(x_{\mathcal{K}_3}, L_{\mathcal{K}_3})$ of respective sphere centers $\mathcal{K}_2$ and $\mathcal{K}_3$ in the corresponding vector basis.

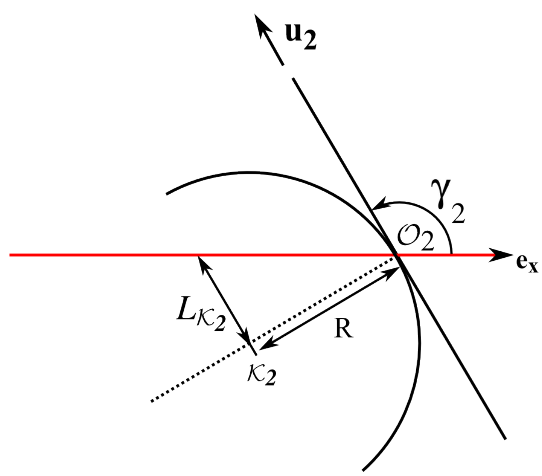

Figure 5: Center position of the spherical mirror 2

As shown on the figure 5 we get:

$$(x_{\mathcal{K}_2}, L_{\mathcal{K}_2}) = \left(L - \frac{R}{\cos(\gamma_2 - \frac{\pi}{2})}, -R\tan(\gamma_2 - \frac{\pi}{2})\right) = \left(L - \frac{R}{\sin(\gamma_2)}, \frac{R}{\tan(\gamma_2)}\right) \tag{15}$$

With the figure 6 we get:

$$(x_{\mathcal{K}_3}, L_{\mathcal{K}_3}) = \left(L + \frac{l}{\tan(\gamma_3)} - \frac{R}{\sin(\gamma_3)}, -\frac{l}{\sin(\gamma_3)} + \frac{R}{\tan(\gamma_3)}\right) \tag{16}$$

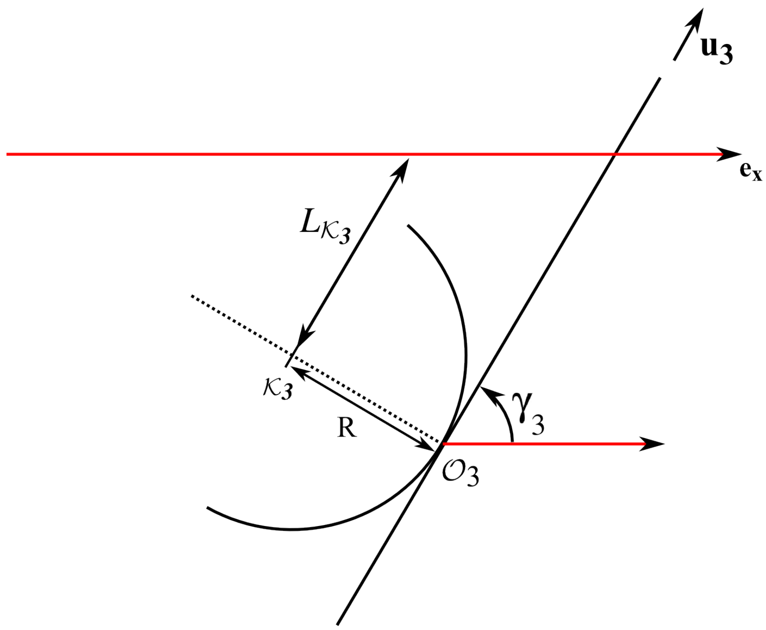

Figure 6: Center position of the spherical mirror 3

2.3.2 Incident beam position on spherical mirrors

We will determine the intersection points $\mathcal{T}'_2$ and $\mathcal{T}'_3$ of incident beams with circle associated to the curved mirror 2 and 3.

The equation of such circles is:

$$\left[(x'_{\mathcal{O}_j} - x_{\mathcal{K}_j})\mathbf{e_x} + (L'_j - L_{\mathcal{K}_j})\mathbf{u_j})\right]^2 = R^2 \tag{17}$$

This give:

$$(x'_{\mathcal{O}_j} - x_{\mathcal{K}_j})^2 + (L'_j - L_{\mathcal{K}_j})^2 + 2(x'_{\mathcal{O}_j} - x_{\mathcal{K}_j})((L'_j - L_{\mathcal{K}_j})\mathbf{u_j} \cdot \mathbf{e_x} = R^2 \tag{18}$$

We get finally:

$$(x'_{\mathcal{O}_j} - x_{\mathcal{K}_j})^2 + (L'_j - L_{\mathcal{K}_j})^2 + 2(x'_{\mathcal{O}_j} - x_{\mathcal{K}_j})(L'_j - L_{\mathcal{K}_j})\cos(\gamma_j) = R^2 \tag{19}$$

We formulate the line equation corresponding to the beam propagation using vector basis $(\mathbf{e_x}, \mathbf{u_j})$:

$$\begin{aligned} x_{\mathcal{O}_i}\mathbf{e_x} + L_i\,\mathbf{u_i} + r\,\mathbf{v_{ij}} &= x_{\mathcal{O}_i}\mathbf{e_x} + L_i\left[\frac{\sin(\gamma_j - \gamma_i)}{\sin(\gamma_j)}\,\mathbf{e_x} + \frac{\sin(\gamma_i)}{\sin(\gamma_j)}\,\mathbf{u_j}\right] \\ &+ r\mu_{ij}\left[\frac{\cos(\alpha_j)}{\sin(\gamma_j)}\,\mathbf{e_x} - \frac{\cos(\alpha_{ir} + \gamma_i)}{\sin(\gamma_j)}\,\mathbf{u_j}\right] \end{aligned} \tag{20}$$

Including equation 20 in the equation 19 we get the equation giving the parameter r'_{ij} for the intersection points $\mathcal{T}'_j$:

$$\begin{aligned} R^2 &= \left[x_{\mathcal{O}_i} + L_i\frac{\sin(\gamma_j-\gamma_i)}{\sin(\gamma_j)} + r'_{ij}\mu_{ij}\frac{\cos(\alpha_j)}{\sin(\gamma_j)} - x_{\mathcal{K}_j}\right]^2 \\ &+ \left[L_i\frac{\sin(\gamma_i)}{\sin(\gamma_j)} - r'_{ij}\mu_{ij}\frac{\cos(\alpha_{ir}+\gamma_i)}{\sin(\gamma_j)} - L_{\mathcal{K}_j}\right]^2 \\ &+ 2\left[x_{\mathcal{O}_i} + L_i\frac{\sin(\gamma_j-\gamma_i)}{\sin(\gamma_j)} + r'_{ij}\mu_{ij}\frac{\cos(\alpha_j)}{\sin(\gamma_j)} - x_{\mathcal{K}_j}\right] \\ &\times \left[L_i\frac{\sin(\gamma_i)}{\sin(\gamma_j)} - r'_{ij}\mu_{ij}\frac{\cos(\alpha_{ir}+\gamma_i)}{\sin(\gamma_j)} - L_{\mathcal{K}_j}\right]\cos(\gamma_j) \end{aligned} \quad (21)$$

We rewrite this equation using quantities a, b, c, d in order to simplify the equation:

$$(a + b\,r'_{ij})^2 + (c + d\,r'_{ij})^2 + 2(a + b\,r'_{ij})(c + d\,r'_{ij})\cos(\gamma_j) = R^2 \quad (22)$$

Then we obtain:

$${r'_{ij}}^2\,(b^2+d^2+2bd\cos(\gamma_j)) + r'_{ij}\,(2ab+2cd+2(ad+bc)\cos(\gamma_j)) + (a^2+c^2+2ac\cos(\gamma_j)) - R^2 = 0 \quad (23)$$

It is a quadratic equation with the form $A{r'_{ij}}^2 + Br'_{ij} + C = 0$. The solution which we are seeking, is at the "right" side of the circle:

$$\boxed{r'_{ij} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}} \quad (24)$$

2.3.3 Reflection on spherical mirrors

The radius joining the center $\mathcal{K}_j$ with the point $\mathcal{T}'_j$ represent the orthogonal to the tangent of the spherical mirror at this point. With the coordinates of these points we get unit vector $\mathbf{u}'^{\perp}_{\mathbf{j}}$ associated to this orthogonal direction as following:

$$\mathbf{u}'^{\perp}_{\mathbf{j}} = \frac{\overrightarrow{\mathcal{K}\mathcal{T}'}}{\|\overrightarrow{\mathcal{K}\mathcal{T}'}\|} = \frac{\overrightarrow{\mathcal{K}\mathcal{T}'}}{R} \quad (25)$$

From the figure 7 we calculate the angular correction γ'_j to make transition from plane mirror to spherical mirror.

$$\mathbf{u}'^{\perp}_{\mathbf{j}} \cdot \mathbf{u_j} = \cos(\frac{\pi}{2} - \gamma'_j) = \sin(\gamma'_j) \quad (26)$$

$$\boxed{\frac{(a + b\,r'_{ij})\cos(\gamma_j) + (c + d\,r'_{ij})}{R} = \sin(\gamma'_j)} \quad (27)$$

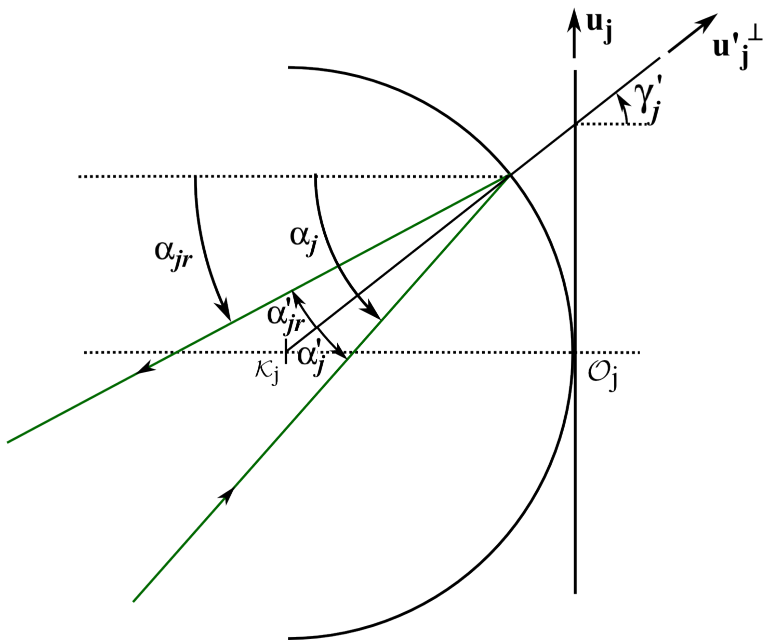

Figure 7: Reflections on the spherical mirror j

We deduce from the figure 7 the equation for reflection law including angular correction due to sphericity:

$$\boxed{\alpha_{jr} = -\alpha_j + 2\gamma'_j} \tag{28}$$

2.3.4 Variables in an equivalent system of mirror parallel to a reference plane mirror

The position along x axis of the equivalent mirror $j = 2, 3$ which pass at the point $\mathcal{T}'_j$ is:

$$\boxed{x'_{\mathcal{O}_j} = x_{\mathcal{O}_i} + L_i \frac{\sin(\gamma_j - \gamma_i)}{\sin(\gamma_j)} + r'_{ij}\, \mu_{ij} \frac{\cos(\alpha_j)}{\sin(\gamma_j)}} \tag{29}$$

In the same way we find the position of the incident beam on the equivalent mirror along the $\mathbf{u_j}$ axis

$$\boxed{L'_j = L_i \frac{\sin(\gamma_i)}{\sin(\gamma_j)} - r'_{ij}\, \mu_{ij} \frac{\cos(\alpha_{ir} + \gamma_i)}{\sin(\gamma_j)}} \tag{30}$$

3 Numerical resolution

3.1 The method and the code

The cavity is assumed to be at the resonance when the propagating beam comes back with the same position and the same incident angle. For the ideal configuration of OMC, the position L_{1reso} and the angle α_{1reso}, which make resonant the beam inside the cavity, are obvious, *i. e.* $L_{1reso} = 0$ and $\alpha_{1reso} = -\gamma_1$. But in the case of random errors for the angular position of mirrors there are no obvious solutions. Since analytical solutions for the beam propagation inside the cavity with four mirrors are too complicated and not relevant for our problem, we will proceed by using an algorithm to find the solution $(L_{1reso}, \alpha_{1reso})$. To help us, previous equations had been established in order to be used iteratively and therefore easily implementable in any codes. Initialization of the algorithm is done by taking the value (L_1, α_1) assuming the position of mirrors 1 would be the ideal configuration for the beam, *i. e.* $(L_1 = 0, \alpha_1 = -\gamma 1 \pm \varepsilon)$ where ε is the angular error. The procedure of the algorithm consist to find roots of the two functions $\delta L_1(L_1, \alpha_1)$ and $\delta\alpha_1(L_1, \alpha_1)$ which are the differences of value L_1 and α_1 before and after a round trip of the beam inside the cavity. Then at each iteration we correct the couple of parameters (L_1, α_1) to have null differences δL_1 and $\delta\alpha_1$. The algorithm has been optimized by using the "good Broyden's method" for finding the two roots of two variables functions.

The program works with an input file of parameters and create an output file with results. Below is shown an example of input file, "`par_cavite.dat`", with a set of parameters , all of them are commented. For simplicity of input file the sign of angular error implemented for each mirror is hard written in the code but can be changed easily before compiling the code again. The compilation is done by writing in a command terminal " `make version11`" after going in the corresponding file, such as
"`version11_recherche_resonnance_optimisee_fonction_erreur_angulaire/`".

```
***************** Cavity parameters ************************
60      Length of cavity in mm
19.209  Distance between two adjacent optical centers in mm
8.876   Angle position of mirror in degree
1499    Radius of curvature for spherical mirrors in mm
0.03    Angular error for mirror positioning in degree
0.0002   First iteration angular step in degree
0.001    First iteration position step in mm
1.E-6   Precision for finding roots (for angles in degree and position in mm)
```

The code is started by typing in a command terminal " `./version11` ", then we obtain this output file "`donnees_sortie_v11.dat`":

```
1 -8.906 0 -0.3776254233884 0.003930800733225
2 -8.9058 0.001 -0.37871115559 0.003932149038560
3 -9.173909486653 -0.347783936083 0.7576565219381 0.004552065445260
```

```
4 -9.232287059091 -0.1151910791433 0.9057057214617 0.005441365775734
5 -8.875212713339 -1.5381011266109 8.339724963990e-05 -2.195989549286e-06
6 -8.875025350805 -1.538231933402 -0.00058988184932 -3.547156279981e-06
7 -8.875054119088 -1.538117252958 -0.00051692015003 -3.108150088859e-06
8 -8.875258058830 -1.537304763700 4.650376093096e-07 2.794956321938e-09


Mirror_Nb  alpha           gamma       L

 1   -8.87525789869126 81.094 -1.53730429866302
 2   -8.93674194116955 98.906 -1.57015133633375
 3   8.81527897825156  81.154 -1.5693452250757
 4   8.87672102174845  98.846 -1.53704663888607
```

The upper part show the different iteration of the program with the iteration numbers in the first column, the angle α_1 in the second one, the position L_1 for the third, and differences δL_1 and $\delta\alpha_1$ for the two last column. The lower part give results for each mirrors respect to the commented column.

3.2 Results

The main idea is to test the tolerancing of construction specifications for the OMC, thus it is essential to take the worst case of angular error configuration. We can find this condition by doing several tests and we get the following sign configuration for the angular error ε_γ:

$$\begin{aligned}\gamma_1 &= \gamma_{10} - \varepsilon_\gamma \text{ or } + \varepsilon_\gamma \\ \gamma_2 &= \gamma_{20} + \varepsilon_\gamma \text{ or } - \varepsilon_\gamma \\ \gamma_3 &= \gamma_{30} + \varepsilon_\gamma \text{ or } - \varepsilon_\gamma \\ \gamma_4 &= \gamma_{40} - \varepsilon_\gamma \text{ or } + \varepsilon_\gamma \end{aligned} \tag{31}$$

where $\gamma_{i0} = 90 \pm 8.876\,^\circ$ correspond to the ideal configuration of OMC. The two possibility of sign configurations are equivalent and give identical results.

The set of parameters taken in the Advanced VIRGO TDR are:

$$\gamma_{i0} = 90 \pm 8.876\,^\circ$$

$$L = 60\,\text{mm} \tag{32}$$

$$l = 19.20\,\text{mm} \tag{33}$$

$$RoC = 1499\,\text{mm} \tag{34}$$

The precision chosen for finding roots is a difference of δL_1 and $\delta\alpha_1$ less than 10^{-6}. Within this configuration we can plot position offset L_{1reso} and tilt α_{1reso} in comparison to the ideal OMC.

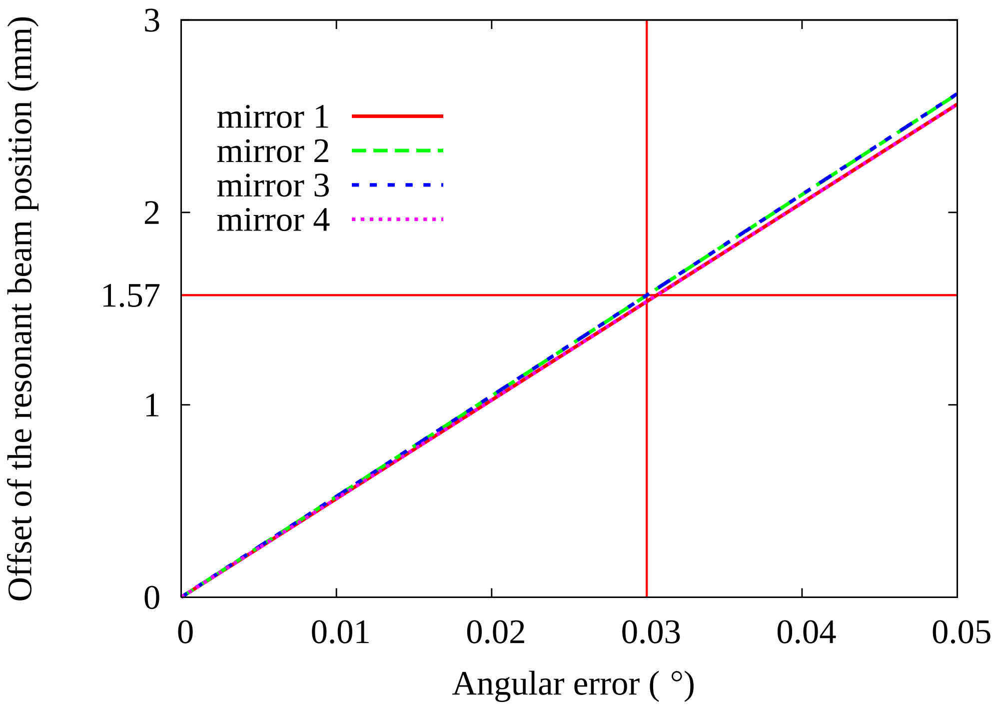

Figure 8: Offset position L_{ireso} as a function of angle error ε_γ.

The first observation of the curve in figure 8 is the offset on each mirror is quite close and strongly linear dependent on the angle error. Assuming the equation $L_{1reso} = a\,\varepsilon_\gamma$ the slope for the two lines are $a = 52.3\,\mathrm{mm}\cdot(^\circ)^{-1}$ for the upper one and $a = 51.2\,\mathrm{mm}\cdot(^\circ)^{-1}$ for the lower one. This linear behaviour is not so surprising since we are dealing with very small angles ε_γ and this has the consequence to make this problem mainly dependent to the first order of ε_γ.

With $\varepsilon_\gamma = 0.03\,^\circ$, which was the specification to be checked, we find an offset $L_{1reso} = 1.57\,\mathrm{mm}$. Then, it is usual to take $2.5\,r$ for calculate the edge of mirrors. Assuming the beam has a radius around $r = 0.3\,\mathrm{mm}$ (the beam waist is 0.256 mm), this constraint corresponds to $1.57 + 2.5\,r = 2.32\,\mathrm{mm}$, which is less than the radius of the polished surface mirror (2.5 mm). These results confirm that an error of $0.03\,^\circ$ is acceptable for the actual configuration of OMC even in the worst case. Similar results had been obtain by Romain Gouaty with a naive model of cavity which confirm the coherence of such results with the code.

In the same idea we can check the angle offset $(\alpha_{ireso} - \gamma_{i0})$ as a function of angle error ε_γ. We observe in figure 9 that the angular offset is also enough small to does not have any

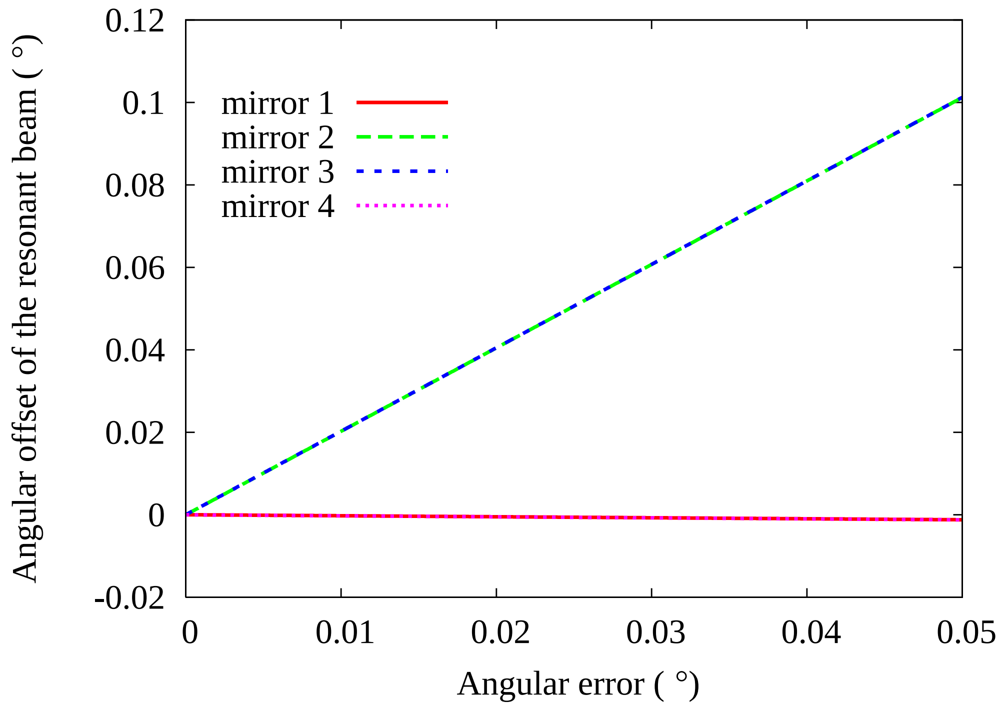

Figure 9: Angle offset $(\alpha_{ireso} - \gamma_{i0})$ as a function of angle error ε_γ.

consequence on the surrounding optical set-up.

4 Conclusion

The tolerancing $\varepsilon_\gamma = \pm 0.03\,°$ confirm to us that the resonant beam does not exit from polished surfaces and does not have any impact on the optical set-up.

The code can be used for other configurations and can be easily modified to adapt it to other kind of cavity such as one spherical mirror for example.

The equations developed in the first part can be used in other contexts and other programming languages.